MATH 1010E University Mathematics
Lecture Notes (week 3)
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1 Left-hand and Right-hand Limits

We define the *left-hand limit* and *right-hand limit* of a function respectively by

$$\lim_{x\to a^-} f(x) := \lim_{\substack{x\to a \\ x<a}} f(x),$$

$$\lim_{x\to a^+} f(x) := \lim_{\substack{x\to a \\ x>a}} f(x).$$

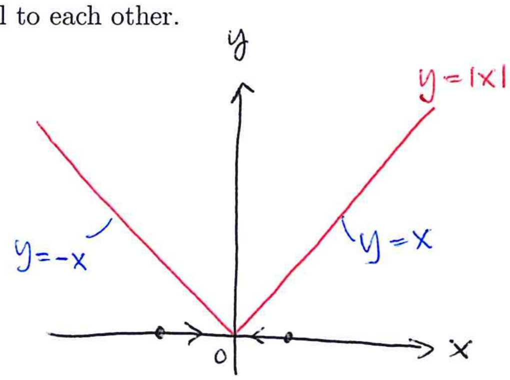
These can be considered as some kind of "half-limits" since we are considering the values $f(x)$ when x is close to a but lying on one side of a. They can be used to prove that a limit exists or not by the following theorem.

Theorem 1.1 *We have the following equivalence:*

$$\lim_{x\to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x).$$

In other words, the limit of $f(x)$ as $x \to a$ exists and equals to L if and only if both the left hand and right hand limits exist and equal to each other.

Example 1.2 *Show that* $\lim_{x\to 0} |x| = 0$.



By definition of absolute value,

$$|x| := \begin{cases} x & \text{when } x \geq 0, \\ -x & \text{when } x < 0. \end{cases}$$

Computing the left-hand and right-hand limits, we get

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0,$$

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0.$$

Therefore, by the theorem above, we have $\lim_{x\to 0} |x| = 0$.

Theorem 1.1 is especially useful when the function is piecewise defined.

Example 1.3 *Calculate the limit* $\lim_{x\to 0} f(x)$ *for the function*

$$f(x) := \begin{cases} x+1 & \textit{when } x > 0, \\ x^2+1 & \textit{when } x < 0. \end{cases}$$

Calculating the left-hand and right-hand limits,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+1) = 1,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1.$$

Since both limits exist and are equal, we have $\lim_{x\to 0} f(x) = 1$.

Example 1.4 *Calculate the limit* $\lim_{x\to 0} f(x)$ *for the function*

$$f(x) := \begin{cases} x+2 & \textit{when } x > 0, \\ x^2+1 & \textit{when } x < 0. \end{cases}$$

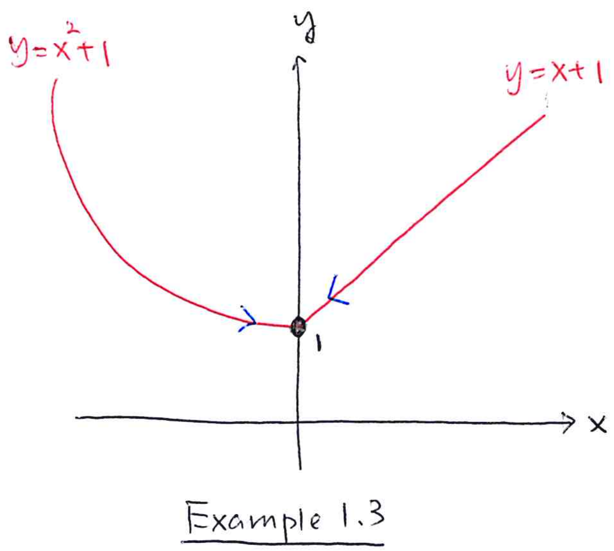
Repeating the calculations as above,

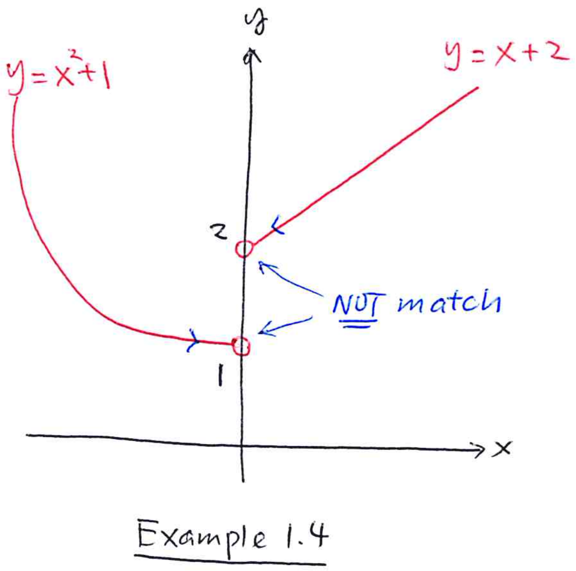
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+1) = 1,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+2) = 2.$$

Although both limits exist, but they are not equal, so the limit $\lim_{x\to 0} f(x)$ does not exists.

The picture shows the difference in their graphs for the two examples above:





2 Limits of Rational Functions as $x \to \infty$

Recall that a rational function is just a quotient of two polynomial functions, that is,

$$f(x) := \frac{p_n(x)}{q_m(x)},$$

where $p_n(x)$, $q_m(x)$ are polynomials of degree n and m respectively.

We look at some examples of limits of rational functions as $x \to \infty$.

Example 2.1 *(i)* $\lim_{x\to\infty} \frac{x^2+2}{x+1} = \lim_{x\to\infty} \frac{x+(2/x)}{1+(1/x)} = \infty$.

(i) $\lim_{x\to\infty} \frac{x+1}{x^2+2} = \lim_{x\to\infty} \frac{(1/x)+(1/x^2)}{1+(2/x^2)} = 0$.

(i) $\lim_{x\to\infty} \frac{2x^2+3}{x^2+2} = \lim_{x\to\infty} \frac{2+(3/x^2)}{1+(2/x^2)} = 2$.

In summary, we have the following:

$$\lim_{x\to\infty} \frac{p_n(x)}{q_m(x)} = \begin{cases} \infty & \text{if } n > m, \\ 0 & \text{if } m > n, \\ \frac{a_n}{b_n} & \text{if } n = m, \end{cases}$$

where the polynomials are $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $q_m(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ with $a_n, b_m \neq 0$.

3 More properties of Limits

Limits behave well with respect to arithmetic operations $\pm, \times, \div$.

Proposition 3.1 *If* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exists, then the limit for* $f(x) \pm g(x)$, $f(x)g(x)$ *and* $f(x)/g(x)$ *all exists as* $x \to a$ *and*

(1) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

(2) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

(3) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *provided that* $\lim_{x\to a} g(x) \neq 0$.

Question: If $\lim_{x\to a}[f(x)\pm g(x)]$ exists, does it mean that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist? Prove it or give a counterexample.

Theorem 3.2 (Comparison Theorem) *If* $f(x) \leq g(x)$ *for all* $x \in (a,b)$ *and* $c \in (a,b)$, *then* $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$ *whenever the limits exist.*

In other words, taking limit preserves the ordering. A useful corollary to the comparison theorem is the following.

Corollary 3.3 (Sandwich Theorem) *Fix some $c \in (a, b)$ and suppose*

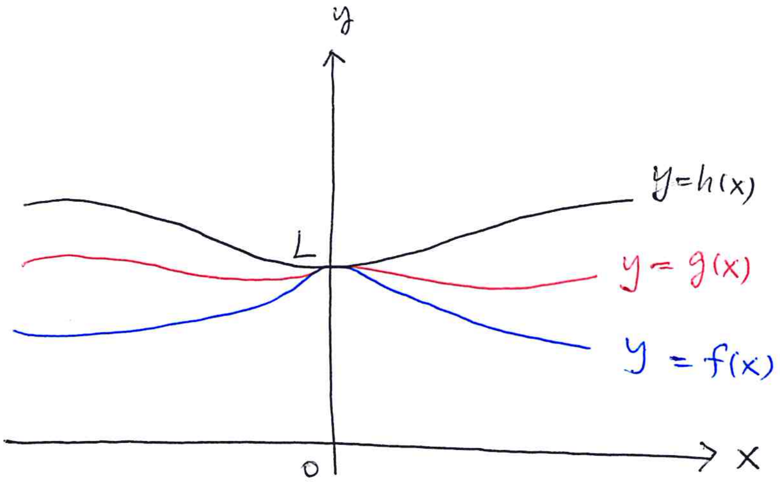
$$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in (a, b).$$

If the limits

$$\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$$

exists and are equal, then we have $\lim_{x \to c} g(x) = L$.

That is, if a function $g(x)$ is "squeezed" between two functions $f(x)$ and $h(x)$ and that these two functions have the same limit as $x \to c$, then the function in the middle will also approach the same limit.



The sandwich theorem is a very useful technique to calculate limits. We will illustrate this by a few examples.

Example 3.4 *Show that $\lim_{x \to 0} \sin x = 0$ by the sandwich theorem.*

Solution: We claim that

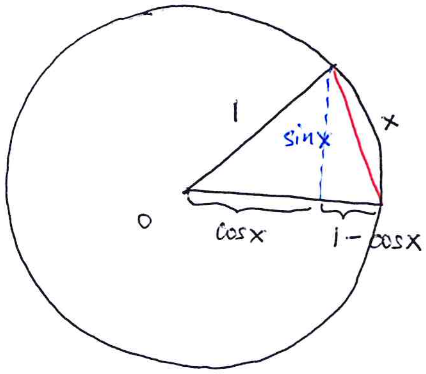
$$-|x| \leq \sin x \leq |x| \qquad \text{for all } x \in \mathbb{R}.$$

If we assume the claim for the moment, since

$$\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|,$$

by the sandwich theorem, it follows that $\lim_{x\to 0} \sin x = 0$.

The inequality we claimed can be proved by the picture below:



The length of the red line segment is shorter than the length of the circular arc joining the same end points. Therefore, we have

$$\sqrt{\sin^2 x + (1 - \cos x)^2} \leq x.$$

It then follows that

$$\sin^2 x \leq \sin^2 x + (1 - \cos x)^2 \leq x^2.$$

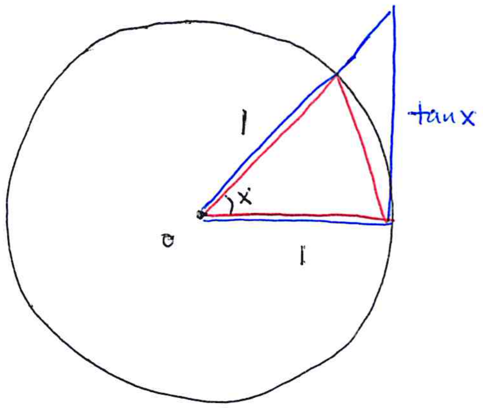
This proves the claim at the very beginning.

Example 3.5 *Show that* $\lim_{x\to 0} \frac{\sin x}{x} = 1$ *using the sandwich theorem.*

Solution: As in Example 3.4, we claim that

$$\cos x < \frac{\sin x}{x} < 1 \qquad \text{for all } x \in \mathbb{R},$$

which implies that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ by the sandwich theorem. To see why the claim is true, again we look at the picture below:



Note that the area of the red triangle is smaller than the area of the sector which is smaller than the area of the blue triangle. Hence we have

$$\frac{1}{2}\sin x \leq \frac{1}{2}x \leq \frac{1}{2}\tan x.$$

The inequality we claimed follows easily from this.
Alternative solution: We can actually use the series definition of $\sin x$ to see why $\lim_{x\to 0}\sin x = 0$. Recall that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Dividing both sides by x, we get

$$\frac{\sin x}{x} = 1 - \frac{x^2}{2!} + \frac{x^4}{5!} - \cdots.$$

If we let $x \to 0$, all the terms involving x on the right hand side goes to 0, therefore, the limit is equal to 1.

Question: Using either the series definition or the double angle formula, prove that

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

What about

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2} = ?$$

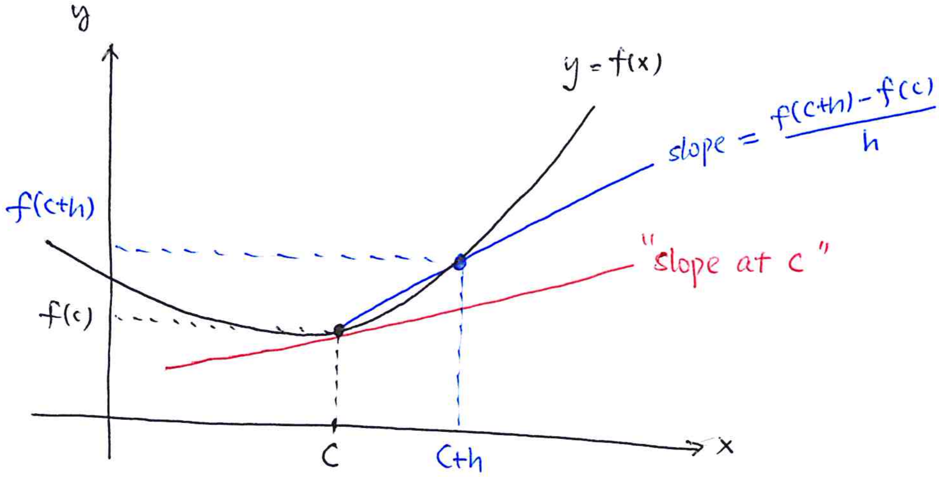
Example 3.6 *By doing a simple change of variable, we can calculate*

$$\lim_{x\to 0}\frac{\sin 2x}{5x} = \lim_{x\to 0}\frac{2}{5}\left(\frac{\sin 2x}{2x}\right) = \frac{2}{5}\lim_{u\to 0}\frac{\sin u}{u} = \frac{2}{5}.$$

Question: What is $\lim_{x\to\infty}\frac{\sin x}{x}$?
(Hint: use the sandwich theorem.)

4 Derivatives as limit of difference quotients

Suppose we are given the graph of a function $y = f(x)$, how do we calculate the slope of the graph at $x = c$?



The slope of the red line is the answer to the question. If we consider the two points on the graph corresponding to $x = c$ and $x = c + h$, the slope of the blue line joining these two points is

$$\left.\frac{\Delta f}{\Delta x}\right|_{x=c} := \frac{f(c+h) - f(c)}{h},$$

which is called the *difference quotient of f at $x = c$.*

Remember that our goal is to calculate the slope of the red line, not the blue line. However, as $h \to 0$, we observe that the blue line should coincide with the red line in the limit. Hence, the limit of the slopes of the blue lines should be the slope of the red line. Therefore, we define

$$f'(c) = \left.\frac{df}{dx}\right|_{x=c} := \lim_{h\to 0} \frac{\Delta f}{\Delta x} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h},$$

which is called the *derivative of f at $x = c$.* In other words, the slope of the graph of f at $x = c$ is given by the number $f'(c)$.

Example 4.1 *Let $f(x) = x$. Calculate the derivative $f'(1)$ from the definition.*

Solution: At $x = 1$, the difference quotient is

$$\left.\frac{\Delta f}{\Delta x}\right|_{x=1} = \frac{f(1+h) - f(1)}{h} = \frac{(1+h) - 1}{h} = 1.$$

Therefore, taking limit as $h \to 0$,

$$f'(1) = \lim_{h\to 0} \left.\frac{\Delta f}{\Delta x}\right|_{x=1} = \lim_{h\to 0} 1 = 1.$$

In fact, we can show that $f'(c) = 1$ for any $c \in \mathbb{R}$.

Example 4.2 *Let $f(x) = x^2$. Calculate the derivative $f'(1)$ from the definition.*

Solution: At $x = 1$, the difference quotient is

$$\left.\frac{\Delta f}{\Delta x}\right|_{x=1} = \frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1^1}{h} = \frac{2h + h^2}{h} = 2 + h.$$

Therefore, taking limit as $h \to 0$,

$$f'(1) = \lim_{h\to 0} \left.\frac{\Delta f}{\Delta x}\right|_{x=1} = \lim_{h\to 0}(2+h) = 2.$$

In fact, we can show that $f'(c) = 2c$ for any $c \in \mathbb{R}$.

From the examples above, we see that we can sometimes calculate the derivative $f'(c)$ at different values of c. Therefore, if we think of c as a variable (defined on where the derivative exists), then we obtain a new function $f'(x)$ which is also called the *derivative of f*. In other words, the derivative of a function is another function (which maybe defined on a smaller domain).

Question: Show that for any $n \in \mathbb{N}$,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

In fact, we will see later that the above formula holds for any $n \in \mathbb{R}$.

5 More complicated examples

In this section, we compute some more complicated derivatives.

Example 5.1 *Let $f(x) = \sqrt{x+2}$, calculate $f'(x)$.*

Solution: The difference quotient is

$$\begin{aligned}
\frac{\Delta f}{\Delta x} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h} \\
&= \frac{(\sqrt{x+h+2}-\sqrt{x+2})(\sqrt{x+h+2}+\sqrt{x+2})}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \frac{(x+h+2)-(x+2)}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}}.
\end{aligned}$$

Taking limit as $h \to 0$, we have

$$f'(x) = \lim_{h\to 0} \frac{\Delta f}{\Delta x} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}.$$

Example 5.2 *Show that*

$$\frac{d}{dx}\sin x = \cos x.$$

Solution: The difference quotient is

$$\begin{aligned}
\frac{\Delta f}{\Delta x} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{\sin(x+h)-\sin x}{h} \\
&= \frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h} \\
&= \sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right).
\end{aligned}$$

Taking limit as $h \to 0$, we have

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\Delta f}{\Delta x} = \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right] = \cos x,$$

using the results from Example 3.5 and the question following it.

Question: Show that

$$\frac{d}{dx}\cos x = -\sin x.$$

6 Differentiability

First, we state precisely the definition of differentiability.

Definition 6.1 *Let $f : (a, b) \to \mathbb{R}$ be a function and fix $c \in (a, b)$. If the limit*

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists (i.e. is a finite real number), then we say that f is differentiable at c *and the value of the limit above is called the* derivative of f at c, *denoted by*

$$f'(c) \quad or \quad \left.\frac{df}{dx}\right|_{x=c}.$$

Remark 6.2 *Note that f has to be defined at $x = c$ to talk about its derivative at $x = c$, this is in contrast with the situation for $\lim_{x \to c} f(x)$, where we do not require f to be defined at $x = c$.*

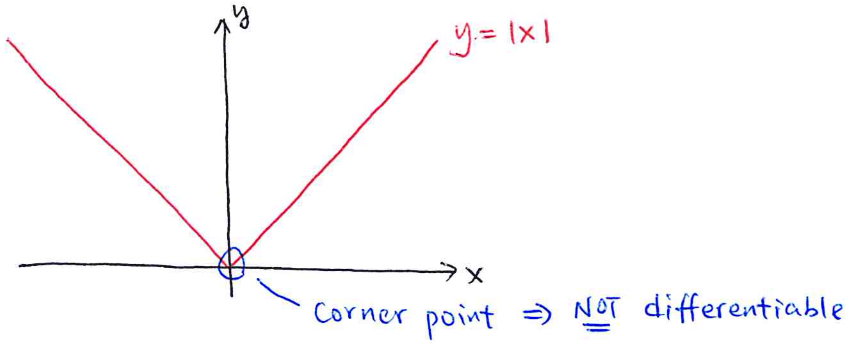
As a matter of fact, not all the functions are differentiable.

Example 6.3 *The function $f(x) = |x|$ is not differentiable at $x = 0$.*

To see this, first we see that the difference quotient is

$$\left.\frac{\Delta f}{\Delta x}\right|_{x=0} = \frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \text{sgn}(h),$$

where sgn is the sign function we saw in last week's lecture. We have seen that $\lim_{h \to 0} \text{sgn}(h)$ does not exist. Therefore, the function f is not differentiable at $x = 0$. If we look at the graph:



There is a sharp corner at $x = 0$. If we approach 0 from the left hand side, the slope should be -1, but if we approach from the right hand side, the slope should be 1. Hence we do not know whether we should assign the slope at $x = 0$ to be 1 or -1. Thus the slope at $x = 0$ is not well-defined.

Question: Show that the function defined below is not differentiable at $x = 0$:

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

Sketch the graph of the function to see why it fails to be differentiable at $x = 0$.

7 Continuity

Now, we define the concept of continuity of a function.

Definition 7.1 *Let $f : (a, b) \to \mathbb{R}$ be a function and fix $c \in (a, b)$. If*

$$\lim_{x \to c} f(x) = f(c),$$

then f is continuous at c.

Remark 7.2 *Note that in order for f to be continuous at c, we indeed require*

(1) f is defined at c,

(2) $\lim_{x \to c} f(x)$ exist, and

(3) the limit is equal to $f(c)$.

Basic examples of continuous functions include all the elementary function: polynomials, trigonometric functions, exponential function and logarithm function, on their domains of definition.

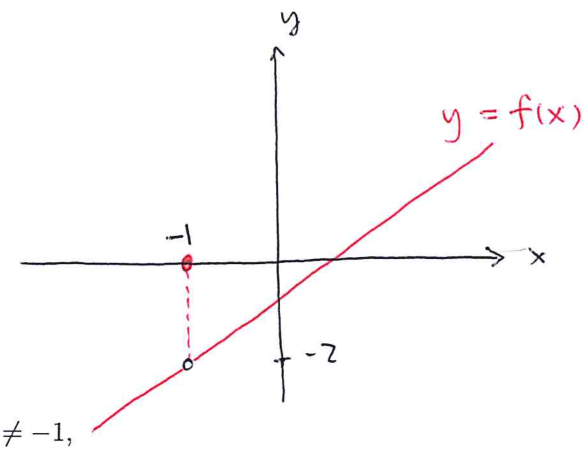
Let's look at some examples of functions which are NOT continuous.

Example 7.3 *The function*

$$f(x) := \begin{cases} \frac{x-1}{x+1} & \text{when } x \neq -1, \\ 0 & \text{when } x = -1. \end{cases}$$

is NOT continuous at $x = -1$ since $\lim_{x \to -1} f(x)$ does not exist:

Example 7.4 *The function*



$$f(x) := \begin{cases} \frac{x^2-1}{x+1} & \textit{when } x \neq -1, \\ 0 & \textit{when } x = -1. \end{cases}$$

is NOT continuous at $x = -1$. *Although this time the limit* $\lim_{x\to -1} f(x)$ *exists but*

$$\lim_{x\to -1} f(x) = -2 \neq 0 = f(-1).$$

Question: How can we change the definition of f to make it continuous? (*Hint: You just have to redefine it at one point!*)

Question: Show that The function

$$f(x) := \begin{cases} x \sin \frac{1}{x} & \text{when } x \neq 0, \\ 0 & \text{when } x = -1. \end{cases}$$

is continuous at $x = 0$.

8 Differentiability vs Continuity

There is a relation between differentiability and continuity of a function.

Theorem 8.1 *If f is differentiable at c, then f is continuous at c.*

Remark 8.2 *The opposite implication is obviously false, i.e. there are functions which is continuous at c but not differentiable at c. For example, $f(x) = |x|$ is continuous at 0 but not differentiable there.*

Proof of Theorem 8.1: Suppose f is differentiable at c. By definition, the limit below exists and is a finite real number

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = f'(c).$$

To show continuity of f at c, we have to prove that

$$\lim_{x\to c} f(x) = f(c),$$

which is equivalent to

$$\lim_{h\to 0} f(c+h) = f(c).$$

To see the last equality,

$$\begin{aligned}
\lim_{h\to 0} f(c+h) &= \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\cdot h + f(c)\right] \\
&= \lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right)\cdot \lim_{h\to 0} h + \lim_{h\to 0} f(c) \\
&= f'(c)\cdot 0 + f(c) = f(c).
\end{aligned}$$

9 Differentiation Rules I

Differentiation behaves well with respect to $\pm$ and multiplication by constants.

Proposition 9.1 *If $f(x)$ and $g(x)$ are differentiable functions, then*

(1) $[f(x)\pm g(x)]' = f'(x)\pm g'(x)$.

(2) $[kg(x)]' = kg'(x)$ *where* $k\in\mathbb{R}$ *is a constant.*

Question: Prove the proposition above starting from the definitions.

Since we have already seen that $(x^n)' = nx^{n-1}$, using Proposition 9.1, we know how to differentiate any polynomials:

$$(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)' = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

We can also use the rules formally to differentiate $\sin x$ and $\cos x$, e.g.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Differentiate term by term, we get

$$\frac{d}{dx}\sin x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \cos x.$$

Question: Using the above idea, show that $\frac{d}{dx}\cos x = -\sin x$ and $\frac{d}{dx}e^x = e^x$.